LINEAR ALGEBRA & CALCULUS

UNIT-I

* MATRICES *

SHORTS (2M)

1. Define linear System of Equation?
2. What is the normal form?
3. Define the rank of the matrix.
4. If the matrix of order (m×n), then that would be the rank of the matrix.
5. Find the rank of the singular matrix of order 4×4
6. What type of the solution exists for $2x+3y=5$, $4x+6y=10$ System?
7. The rank of 2×2 matrix with all elements are 3.
8. Write the condition for the homogeneous System of Equations possess trivial Solutions.
9. Define the Echelon form of a matrix. what is the rank of a matrix which is in Echelon form?
10. Define a Singular matrix. What is the inverse of a Singular matrix.

1. Ans :

Linear System of Equation :

An Equation of the form $a_1x_1 + a_2x_2 + \dots + a_nx_n = b$ where $x_1, x_2, \dots x_n$ are unknowns and $a_1, a_2, \dots a_n, b$ are constants is called a linear Equation in 'n' unknowns.

Consider the system of m Eqn in 'n' unknowns as :

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_2 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \dots \dots \dots \dots \\ \dots \dots \dots \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{array}\right\}$$

where a's and $b_1, b_2 \dots b_n$ are constant. An ordered n-tuple $(x_1, x_2, x_3) \cdot x_n)$ satisfying all the Eqn in above Eqn is called a Solution of the System.

2. Ans :

Normal form :

Every $m \times n$ matrix of rank 'r' can be reduced to the form I_r, $[I_r, 0]$ or $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ by the finite chain of elementary row and Column Operations, where I_r is the r-rowed unit matrix. The above form is called "normal matrix" or "first canonical form" of the matrix.

3. Ans :

Rank of matrix :

Let A be an $m \times n$ matrix. If A is a null matrix, we define its rank to be 0 (zero).

If A is a non-zero matrix, we say that r is the rank of A if

(i) every $(r+1)^{th}$ order minor of A is 0 (zero) and

(ii) there exists at least one r^{th} order minor of A which is not zero

Rank of A is denoted by $\rho(A)$.

4. Ans:

The rank of a unit matrix of order $m \times n$. If a matrix A is of order $m \times n$, then $\rho(A) \le \min\{m,n\}$ = minimum of m, n.

If A is of order $n \times n$ and $|A| \neq 0$, then the rank of $A = n$. If A is order $n \times n$ and $|A| = 0$, then the rank of A will be less than n.

5. Ans:

A sigular matrix is an arrangement of elements in row and columns in a square form, whose determinant is 0 (zero).

All the rows and columns are not unique. So, the least number of unique rows and columns determines the rank of the matrix.

Assume, there is 4×4 singular matrix, then its rank should be less than or equal to 4.

Hence, the rank of singular matrix is less than number of rows/column.

6. Ans: Given Equation $2x + 3y = 5$ and $4x + 6y = 10$

Equation of the form $AX = B$

$$\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 3 & 5 \\ 4 & 6 & 10 \end{bmatrix}$$

$\Rightarrow R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 2 & 3 & 5 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow 2x + 3y = 5$$

$$2x = \frac{5 - 3y}{2}$$

$$\Rightarrow 2\left(\frac{5-3}{2}\right)y + 3y = 5$$

$$2y + 3y = 5$$

$$\boxed{y = 5}$$

$$\boxed{x = -5}$$

8. Ans:

For homogeneous Systems we have the following conditions:

A $n \times n$ homogeneous System of linear Equations has a Unique Solution (the trivial Solution) if and only if its determinant is non-zero. If this determinant is zero, then the System has an infinite number of solutions. or non-trivial Solutions.

9. Ans:

Echelon Form of a matrix:

A matrix is Said to be in Echelon form if it has the following properties

i, Zero rows, if any, are below any non-zero row.

(ii) The first non-zero entry in each non-zero row is Equal to 1.

(iii) The no. of zero's before the first non-zero element in a row is Less than the no. of Such Zeros in the next row.

This is the formation of Echelon form $\begin{bmatrix} 2 & 5 & 3 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$

10. Ans :

Singular matrix :

A singular matrix is a Square matrix whose determinant is zero. Since the determinant is zero, a Singular matrix is non - invertible , which does not have an inverse .

* EIGEN VALUES, EIGEN VECTORS & ORTHOGONAL TRANSFORMATION *

SHORTS (2M)

1. If λ is the Eigen value of a matrix A, then prove that $k\lambda$ is the Eigen Value of KA.

2. State Cayley - Hamilton theorem. How do you find the inverse of a matrix by Using Cayley - Hamilton theorem?

3. Find the matrix Corresponding to quadratic form $x^2 + 4xy + 2y^2$.

4. Find the Sum of the Eigen Values of matrix $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

5. Find the product of the Eigen values of $\begin{bmatrix} 2 & 2 \\ 2 & 6 \end{bmatrix}$

6. Find the Eigen Vector Corresponding to $\lambda = 5$ for the matrix $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

7. If 5 is an Eigen value of A then find the Eigen Value of $4A + 5I$.

8. Write the quadratic form associated with $\begin{bmatrix} 4 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$

9. Find the nature of the quadratic form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

10. Find the Eigen Values of A^T. If 1 and 2 are the Eigen values of A.

1. Ans:

let A be an $n \times n$ matrix. Let K be an eigen vector of A corresponding to the eigen value λ.

Then by definition $KA = K\lambda$

i.e., $KA = \lambda I K$

$KA - \lambda I K = 0$

$(A - \lambda I)K = 0$

This will have a non-zero solution K, if and only if $|A - \lambda I| =$

2 Ans:

Cayley – Hamilton theorem:

Every square matrix satisfies its own characteristic equation.

A^{-1} using cayley - Hamilton theorem:

i.e., $(-1)^n [A^n + a_1 A^{n-1} + a_2 A^{n-2} + \ldots + a_n I] = 0.$

$\Rightarrow A^n + a_1 A^{n-1} + a_2 A^{n-2} + \ldots + a_n I = 0.$

$\Rightarrow A^{-1} [A^n + a_1 A^{n-1} + a_2 A^{n-2} + \ldots + a_n I] = 0$

if A is a non-singular, then we have

$a_n A^{-1} = -A^{n-1} - a_1 A^{n-2} \ldots - a_{n-1} I$

$A^{-1} = \left(-\frac{1}{a_n}\right) [A^{n-1} + a_1 A^{n-2} + \ldots + a_{n-1} I]$

3. Ans:

The given Quadratic form can be written as

$$(x\ y)\begin{bmatrix} x^2 & \frac{xy}{2} \\ \frac{yx}{2} & y^2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

i.e. $$(x\ y)\begin{bmatrix} 1 & \frac{4}{2} \\ \frac{4}{2} & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$\therefore$ The Corresponding matrix is $\begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$ //

4. Ans Given $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

Sum of the eigen value = Trace of A

= Sum of Diagonal

= 1+4

= 5

5. Ans Given $A = \begin{bmatrix} 2 & 2 \\ 2 & 6 \end{bmatrix}$

product of the eigen value = |A|

= (12-4)

= 8.

6 Ans:

The Characteristic Equation $A = \begin{bmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 4 \\ 0 & 0 & 5-\lambda \end{bmatrix}$

Given $\lambda = 5$, $\Rightarrow \begin{bmatrix} 3-5 & 1 & 4 \\ 0 & 2-5 & 4 \\ 0 & 0 & 5-5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} -2 & 1 & 4 \\ 0 & -3 & 4 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -2x_1 + x_2 + x_3 = 0$$

$$\Rightarrow -3x_2 + 4x_3 = 0$$

Let $x_3 = k$,

$$\Rightarrow -3x_2 + 4k = 0$$

$$\Rightarrow x_2 = \frac{4}{3}k$$

$$\Rightarrow -2x_1 + \left(\frac{4}{3}k\right) + k = 0$$

$$x_1 = \frac{14}{3}k$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 14/3\,k \\ 4/3\,k \\ k \end{bmatrix}$$

8. Ans:

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $A = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$, $X^T = [x \ y \ z]$

$\therefore$ Required Quadratic from $= X^T A X = [x \ y \ z] \begin{bmatrix} 4 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$= [x \ y \ z] \begin{bmatrix} 4x + y + 2z \\ x + 2y + 3z \\ 2x + 3y + z \end{bmatrix}$

$= x(4x + y + 2z) + y(x + 2y + 3z) + z(2x + 3y + z)$

$= 4x^2 + 2y^2 + z^2 + 2xy + 4xz + 6yz$ //

9. Ans:

The characteristic Equation of A is $|A - \lambda I| = 0$

i.e. $\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 4-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)[(4-\lambda)(3-\lambda) - 0] - 0[\ ] + 0[\ ] = 0$

$\Rightarrow (1-\lambda)[\lambda^2 - 7\lambda + 12] = 0$

$\Rightarrow (1-\lambda)(\lambda - 4)(\lambda - 3) = 0$

$\Rightarrow \lambda = 1, 4, 3$

$\therefore$ The Eigen values of given ~~matrix~~ is positive, the nature of the given quadratic form is positive definite.

10. Ans:

The characteristic polynomial $P_A(t) = \det(A - tI)$ of A is the same as the characteristic polynomial $P_{A^T}(t) = \det(A^T - tI)$ of the transpose A^T.

We have $P_{A^T}(t) = \det(A^T - tI)$

$= \det(A^T - tI^T)$

$= \det((A - tI)^T)$

$= \det(A - tI)$

$= P_A(t)$

$\because$ since $I^T = I$, $\det(B^T) = \det(B)$ for any square matrix B.

$\therefore$ We obtain $P_{A^T}(t) = P_A(t)$ and the eigen values of A and A^T are same.